"Multiple Linear Regression"

Multiple Linear regression model is a versatile statistical model for evaluating relationships between a continuous target and predictors. It involves multiple predictors or independent variables & dependent variable.

On the contrary in simple linear regression, we have one dependent and one independent variable.

The represented expression of multiple linear regression would be

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \epsilon$$

where,

Y = Dependent variable (target)

$x_1, x_2, \ldots, x_n$ are independent variables (features).

β_0 = intercept

$\beta_1, \beta_2, \beta_3, \ldots, \beta_n$ = Coefficient of independent variables.

ϵ = Error term.

Let's consider an example

x_1	x_2	Y
1	4	1
2	5	6
3	8	8
4	2	12

The above table can be written as matrix

$$X = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 5 \\ 1 & 3 & 8 \\ 1 & 4 & 2 \end{bmatrix} \quad \& \quad Y = \begin{bmatrix} 1 \\ 6 \\ 8 \\ 12 \end{bmatrix}$$

ias.

The coefficient of multiple regression equation is given as

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} \begin{array}{l} \rightarrow \text{Intercept} \\ \rightarrow \text{Ind. var 1 } (x_1) \text{ (coef)} \\ \rightarrow \text{Ind. var 2 } (x_2) \text{ (coef)} \end{array}$$

$$\hat{\beta} = \left(\left(X^T X\right)^{-1} X^T\right) Y$$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 5 & 8 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 5 \\ 1 & 3 & 8 \\ 1 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 & 19 \\ 10 & 30 & 46 \\ 19 & 46 & 109 \end{bmatrix}$$

(2)

Let say $A = (X^T X)^{-1} = \begin{bmatrix} 4 & 10 & 19 \\ 10 & 30 & 46 \\ 19 & 46 & 109 \end{bmatrix}^{-1}$

$\therefore \det(A) = 4(30\times109 - 46\times46)$
$- 10(10\times109 - 46\times19) +$
$19(10\times46 - 30\times19)$
$= 36$

Find the matrix minors:

$a_{11}=4, \; M_{11} = \begin{vmatrix} 30 & 46 \\ 46 & 109 \end{vmatrix} = 1154$

$a_{12}=10, \; M_{12} = \begin{vmatrix} 10 & 46 \\ 19 & 109 \end{vmatrix} = (10\times109) - (46\times19) = 216$

$a_{13}=19, \; M_{13} = \begin{vmatrix} 10 & 30 \\ 19 & 46 \end{vmatrix} = -110$

$a_{21}=10, \; M_{21} = \begin{vmatrix} 10 & 19 \\ 46 & 109 \end{vmatrix} = 216$

$a_{22}=30 \; M_{22} = \begin{vmatrix} 4 & 19 \\ 19 & 109 \end{vmatrix} = 75$

$a_{23}=46, M_{23}=-6$; $a_{31}=19, M_{31}=-110$

$a_{31}=19, M_{31}=-110$; $a_{32}=46, M_{32}=-6$

$a_{33}=109, M_{33}=20$

check board sign pattern. $\begin{bmatrix} +,-,+,-,+ \\ -,+,-,+ \end{bmatrix}$

$\therefore$ Cofactor matrix $= \begin{bmatrix} 1154 & -216 & -110 \\ -216 & 75 & 6 \\ -110 & 6 & 20 \end{bmatrix}$

$Adj(A) = \begin{bmatrix} 1154 & -216 & -110 \\ -216 & 75 & 6 \\ -110 & 6 & 20 \end{bmatrix}$

Now multiply by $\frac{1}{\det(A)} = \frac{1}{366}$

$\therefore A^{-1} = \begin{bmatrix} \frac{1154}{366} & \frac{-216}{366} & \cdots \\ \cdots & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 3.15 & -0.59 & -0.30 \\ -0.59 & 0.20 & 0.02 \\ -0.30 & 0.02 & 0.05 \end{bmatrix}$

How find inverse of a matrix?

Given matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Step 1: check if the matrix is invertible

For example, A to have an inverse, its determinant must be non-zero. The determinant det(A) is calculated as,

$\det(A) = ad - bc$.

If $\det(A)=0$; then matrix is not invertible.

Step 2: If $\det(A) \neq 0$ the inverse of A is given by

$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Step 3: Substitute the values into formula

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the inverse is

$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} b & -b \\ -c & a \end{bmatrix}$

Example: $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$\det(A) = (4)(1) - (3)(2) = -2$

$\det(A) = -2$, the inverse exists.

$\therefore A^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$

$\therefore \; (X^T X)^{-1} X^T$

$$= \begin{bmatrix} 3.15 & -0.59 & -0.30 \\ -0.59 & 0.20 & 0.016 \\ -0.30 & 0.016 & 0.054 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 5 & 8 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.05 & 0.47 & -1.02 & 0.19 \\ -0.32 & -0.098 & 0.155 & 0.26 \\ -0.065 & 0.005 & 0.185 & -0.125 \end{bmatrix}$$

$$\therefore \; \beta = ((x^T x)^{-1} x^T) Y$$

$$= \begin{bmatrix} 0.05 & 0.47 & -1.02 & 0.19 \\ -0.32 & -0.098 & 0.155 & 0.26 \\ -0.065 & 0.005 & 0.185 & -0.125 \end{bmatrix} \times \begin{bmatrix} 1 \\ 6 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 1.69 \\ 3.48 \\ -0.05 \end{bmatrix} \begin{matrix} \rightarrow \beta_0 \\ \rightarrow \beta_1 \\ \rightarrow \beta_2 \end{matrix}$$

here, $\beta_0 = -1.69$, $\beta_1 = 3.48$, $\beta_2 = -0.05$

$$\therefore \; Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$$= -1.69 + 3.48 x_1 + (-0.05) x_2$$

$$= -1.69 + 3.48 x_1 - 0.05 x_2$$

(This is the constructed model)