Exercises on SVMs

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SVM decision formula is as below :

$$f(x) = w^T x + b$$

Problem 1 — classify a point

Given: $w = [2, -1], \ b = -0.5, \ x = [1, 2]$.
Formula: $f(x) = w^T x + b$.

Solution

1. Compute $w^T x = 2 \cdot 1 + (-1) \cdot 2 = 2 - 2 = 0$.
2. $f(x) = 0 + (-0.5) = -0.5$.
3. Decision: $\text{sign}(f(x)) = \text{sign}(-0.5) = -1$.
   Answer: $f(x) = -0.5$, classified as **-1**.

Problem 2 — distance from point to hyperplane

Given: $w = [3, 4], \ b = -6, \ x = [2, 1]$.
Formula: $f(x) = w^T x + b$. Distance to hyperplane = $\frac{|f(x)|}{\|w\|}$.

Solution

1. $w^T x = 3 \cdot 2 + 4 \cdot 1 = 6 + 4 = 10$.
2. $f(x) = 10 + (-6) = 4$.
3. $\|w\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

4. Distance = $\frac{|4|}{5} = 0.8$.
5. Decision: $\mathrm{sign}(4) = +1$.

   Answer: $f(x) = 4$, classified **+1**, distance to hyperplane = **0.8**.

Problem 3 — compute margin (canonical SVM)

Given: For a canonical SVM (support vectors satisfy $y_i f(x_i) = \pm 1$) with $w = [1,\ 1]$. (No bias needed for margin formula here.)

Formula: margin = $\frac{2}{\|w\|}$. Also $f(x) = w^T x + b$ (b not used for margin numeric here).

Solution

1. $\|w\| = \sqrt{1^2 + 1^2} = \sqrt{2}$.
2. margin = $\frac{2}{\sqrt{2}} = \sqrt{2} \approx 1.4142$.

   Answer: margin = $\sqrt{2}$ ($\approx$ **1.414**).

Problem 4 — identify support vectors

Given: $w = [1,\ -1],\ \ b = 0$. Points: $p_1 = [1, 0],\ \ p_2 = [0, 1],\ \ p_3 = [2, 1]$.
Formula: $f(x) = w^T x + b$. Support vectors (in canonical scaling) are those with $|f(x)| = 1$.

Solution

- For $p_1 = [1, 0]$: $w^T p_1 = 1 \cdot 1 + (-1) \cdot 0 = 1$. So $f(p_1) = 1 \to |f| = 1 \Rightarrow$ **support vector** (class +1).
- For $p_2 = [0, 1]$: $w^T p_2 = 0 + (-1) \cdot 1 = -1$. So $f(p_2) = -1 \to |f| = 1 \Rightarrow$ **support vector** (class -1).
- For $p_3 = [2, 1]$: $w^T p_3 = 2 + (-1) \cdot 1 = 1$. So $f(p_3) = 1 \to |f| = 1 \Rightarrow$ **support vector** (class +1).

  Answer: All three points have $|f| = 1$ and are support vectors.

Problem 5 — detect misclassification and margin check

Given: $w = [-1,\ 2],\ \ b = 0.5$. Two labeled points:

- $x^{(1)} = [1, 1]$ with label $y^{(1)} = +1$,
- $x^{(2)} = [2, 0]$ with label $y^{(2)} = +1$.

  Formula: $f(x) = w^T x + b$. A point is correctly classified if $y \cdot f(x) > 0$. If using canonical margin check, margin threshold is 1 (i.e., $\mid f(x) \mid \ \geqslant 1$ is outside/on margin; $\mid f(x) \mid \ < 1$ is inside margin).

Solution

- For $x^{(1)} = [1, 1]$:
  $w^T x^{(1)} = -1 \cdot 1 + 2 \cdot 1 = -1 + 2 = 1.$
  $f(x^{(1)}) = 1 + 0.5 = 1.5.$
  $y^{(1)} f(x^{(1)}) = (+1) \cdot 1.5 = 1.5 > 0 \rightarrow$ **correctly classified**. Also $\mid f \mid \ = 1.5 \geqslant 1$ (outside margin).
- For $x^{(2)} = [2, 0]$:
  $w^T x^{(2)} = -1 \cdot 2 + 2 \cdot 0 = -2 + 0 = -2.$
  $f(x^{(2)}) = -2 + 0.5 = -1.5.$
  $y^{(2)} f(x^{(2)}) = (+1) \cdot (-1.5) = -1.5 < 0 \rightarrow$ **misclassified**. Also $\mid f \mid \ = 1.5 \geqslant 1$ but sign is wrong so misclassification (on the wrong side of hyperplane).

Answer: $x^{(1)} \rightarrow f\ = 1.5 \rightarrow$ correct (+1). $x^{(2)} \rightarrow f\ = -1.5 \rightarrow$ **misclassified** (should be +1 but predicted -1).

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