@SSRoy

NONLINEAR REGRESSION

A Short Tutorial with a Solved Example

1. What Nonlinear Regression Is

Linear regression fits a model that is a straight-line combination of the parameters. **Nonlinear regression** fits a model in which the response depends on the parameters in a **nonlinear** way. The general form is:

$$Y = f(X, \boldsymbol{\theta}) + \varepsilon$$

where Y is the response, X the predictor, θ the parameters to be estimated, f a nonlinear function, and ε the random error. Two common examples:

$$Y = \theta_1\, e^{\theta_2 X} + \varepsilon$$

$$Y = \theta_1\left(1 - e^{-\theta_2 X}\right) + \varepsilon$$

The first is exponential growth/decay; the second is a saturating "growth curve" that rises quickly then levels off.

2. Why It Is Needed

Many real relationships are simply not straight lines. Growth, decay, saturation, and dose–response effects all bend. Forcing a straight line onto such data gives poor predictions and can produce impossible values. A nonlinear model matches the true shape of the data, so it predicts better and its parameters carry real physical meaning (a growth rate, a maximum limit, and so on).

3. How It Is Fitted

The principle is the same as least squares: choose the parameters that minimise the sum of squared residuals. The difference is that the normal equations are now **nonlinear** and usually have no closed-form solution, so they are solved by **iterative numerical methods** (for example Gauss–Newton), starting from initial guesses and improving them step by step.

For some models a shortcut exists: a suitable **transformation** turns the nonlinear model into a linear one, which can then be fitted by ordinary least squares. The solved example below uses this approach.

4. Solved Example — Predicting Concrete Compressive Strength

Concrete gains strength with age, quickly at first and then levelling off — a saturating curve, not a straight line. A suitable nonlinear model relating strength S (MPa) to age t (days) is:

$$S = \frac{a\,t}{b + t}$$

where a is the long-term maximum strength and b controls how fast it is approached. The observed data (a small representative sample in the spirit of the UCI Concrete dataset, Yeh 1998):

Age t (days)	Strength S (MPa)
3	11.5
7	20.5
14	29.0
28	36.5
90	45.0

Step 1 — Linearise the model

Taking reciprocals of both sides turns the curve into a straight line:

$$\frac{1}{S} = \frac{b}{a} \cdot \frac{1}{t} + \frac{1}{a}$$

This has the linear form:

$$v = m\,u + c, \qquad u = \frac{1}{t}, \quad v = \frac{1}{S}$$

So plotting v = 1/S against u = 1/t gives a straight line with slope m = b/a and intercept c = 1/a.

Step 2 — Transform the data

t	S	u = 1/t	v = 1/S
3	11.5	0.3333	0.0870
7	20.5	0.1429	0.0488
14	29.0	0.0714	0.0345
28	36.5	0.0357	0.0274
90	45.0	0.0111	0.0222

Sums (n = 5): Σu = 0.5944, Σv = 0.2198, Σu^2 = 0.13802, Σuv = 0.039643, $\bar{u}$ = 0.11888, $\bar{v}$ = 0.04397.

Step 3 — Least-squares slope and intercept

$$m = \frac{\sum u_i v_i - \left(\sum u_i\right)\left(\sum v_i\right)/n}{\sum u_i^2 - \left(\sum u_i\right)^2/n} = \frac{0.013506}{0.06735} = 0.2005$$

$$c = \bar{v} - m\,\bar{u} = 0.04397 - 0.2005(0.11888) = 0.02013$$

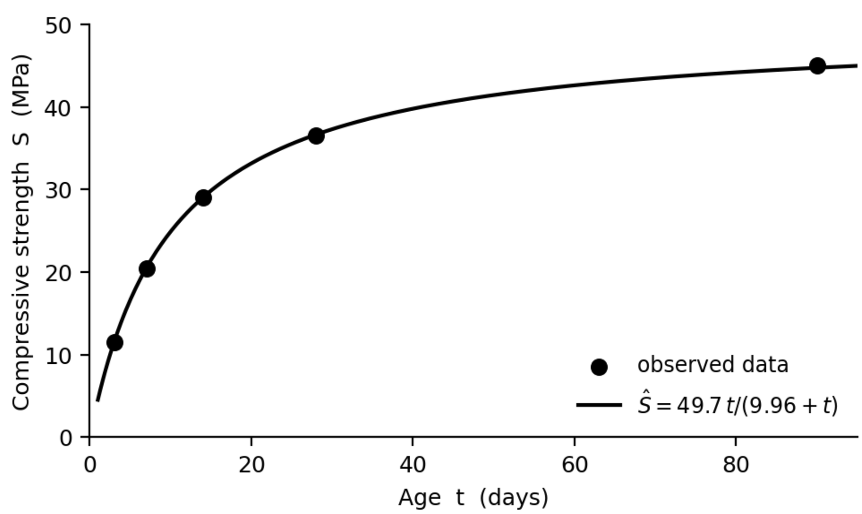
Step 4 — Recover the original parameters

Since c = 1/a and m = b/a:

$$a = \frac{1}{c} = \frac{1}{0.02013} \approx 49.7, \qquad b = m\,a = 0.2005 \times 49.7 \approx 9.96$$

So the fitted nonlinear model is:

$$\hat{S} = \frac{49.7\,t}{9.96 + t}$$



Fitted saturating curve through the concrete strength–age data.

Step 5 — Predict

Predicted compressive strength at t = 28 days:

$$\hat{S} = \frac{49.7 \times 28}{9.96 + 28} = \frac{1391.6}{37.96} \approx 36.7 \text{ MPa}$$

Result: the model predicts about 36.7 MPa at 28 days, matching the observed 36.5 MPa. The fitted a ≈ 49.7 MPa is the estimated long-term maximum strength — a physically meaningful quantity that a straight line could never provide.

5. Key Points

- Nonlinear regression fits models where the response depends nonlinearly on the parameters.
- It is needed when data bend — growth, decay, or saturation — which straight lines cannot capture.
- It is fitted by least squares, usually solved iteratively; some models linearise via a transformation.
- Its parameters often have direct physical meaning (here, the maximum strength a).

Reference: N. R. Draper and H. Smith, Applied Regression Analysis, 3rd ed., Chapter 24 (Nonlinear Estimation). Dataset in the spirit of the UCI Concrete Compressive Strength data (I.-C. Yeh, 1998).