MULTIPLE LINEAR REGRESSION — WORKED NUMERICAL

Using $b = (X'X)^{-1}X'Y$

Problem. Fit the two-predictor model $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$ to the three observations below, using the matrix formula $b = (X'X)^{-1}X'Y$.

Obs	X_1	X_2	Y
1	−1	−1	2
2	0	2	5
3	1	−1	8

Step 1 — Build X and Y

The first column of X is all 1s, for the intercept; the next two columns are the predictor values:

$$\mathbf{X} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

Step 2 — Form X′X

The off-diagonal entries are 0 because the columns are orthogonal ($\Sigma X_1 = 0$, $\Sigma X_2 = 0$, $\Sigma X_1 X_2 = 0$) — giving the simplest possible case, a diagonal matrix:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Step 3 — Form X′Y

The entries are ΣY, $\Sigma X_1 Y$ and $\Sigma X_2 Y$:

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} 15 \\ 6 \\ 0 \end{bmatrix}$$

Step 4 — Invert X′X

For a diagonal matrix, the inverse is simply the reciprocal of each diagonal entry:

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

Step 5 — Compute b

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = \begin{bmatrix} 15/3 \\ 6/2 \\ 0/6 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$$

Result

$b_0 = 5$, $b_1 = 3$, $b_2 = 0$, so the fitted model is:

$$\hat{Y} = 5 + 3X_1 + 0 \cdot X_2 = 5 + 3X_1$$

Check: Obs 1: $5 + 3(-1) = 2$ ✓ Obs 2: $5 + 3(0) = 5$ ✓ Obs 3: $5 + 3(1) = 8$ ✓ All three points fit exactly. The coefficient $b_2 = 0$ shows X_2 adds nothing once X_1 is in the model.